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Mh4714 Week 10

Week 10

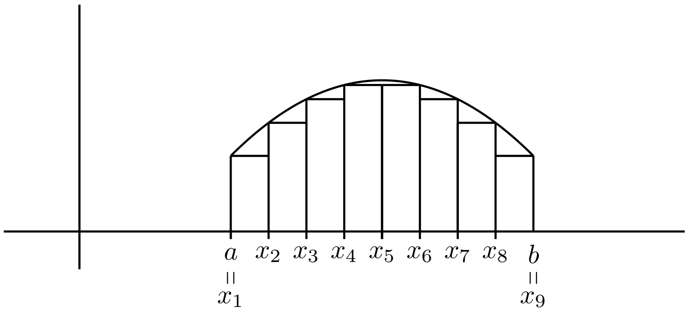
0.1 Integration

Lower Sums
Let f be a function bounded over the interval $[a, b]$ with $f(x) > 0, \forall x \in [a, b]$. We can attempt to underestimate the area between the graph of f, the x-axis and the lines $y = a, y = b$ by by marking off $n+1$ points $x_0 = a, x_1, \dots, x_n = b$ which partitions the interval into n sub-intervals and constructing a rectangle on each sub-interval $[x_{i-1}, x_i], i = 1, 2, \dots n$ with height equal to the greatest lower bound of $\{f(x) : x \in [x_{i-1}, x_i]\}, i = 1, 2, \dots n$.

If we let $m_i = \text{glb}\{f(x) : x \in [x_{i-1}, x_i]\}, i = 0, 1, \dots n$ and $P = \{x_0, x_1, \dots, x_n\}$ then we define:

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$L(f, P)$ is the sum of the areas of the underestimating rectangles and is called a *lower sum*.
Underestimation:



Upper Sums

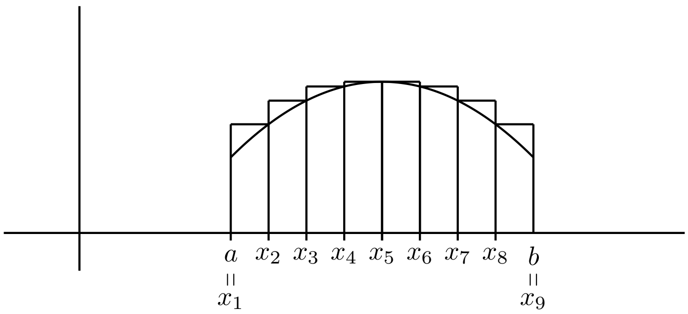
We can attempt to overestimate the area between the graph of f, the x-axis and the lines $y = a, y = b$ by by marking off $n+1$ points $x_0 = a, x_1, \dots, x_n = b$ which partitions the interval into n sub-intervals and constructing a rectangle on each sub-interval $[x_{i-1}, x_i], i = 1, 2, \dots n$ with height equal to the least upper bound of $\{f(x) : x \in [x_{i-1}, x_i]\}, i = 1, 2, \dots n$.

If we let $M_i = \text{lub}\{f(x) : x \in [x_{i-1}, x_i]\}, i = 0, 1, \dots n$ then we define:

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$U(f, P)$ is the sum of the areas of the overestimating rectangles and is called an *upper sum.*

Overestimation:



The set of points $P = \{x_0 = a, x_1, \dots, x_n = b\}$ is called a *partition* of the interval $[a, b]$.

We can add points to any partition to make it *finer.*

It is easy to show that if we make any partition finer then the lower sum gets bigger and the upper sum gets smaller.

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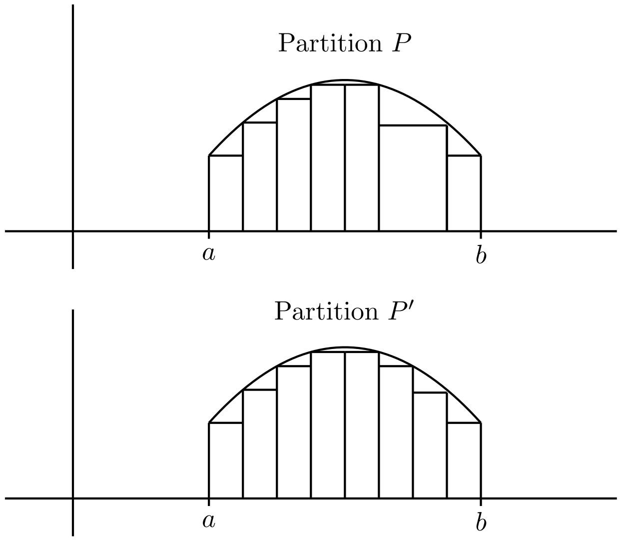
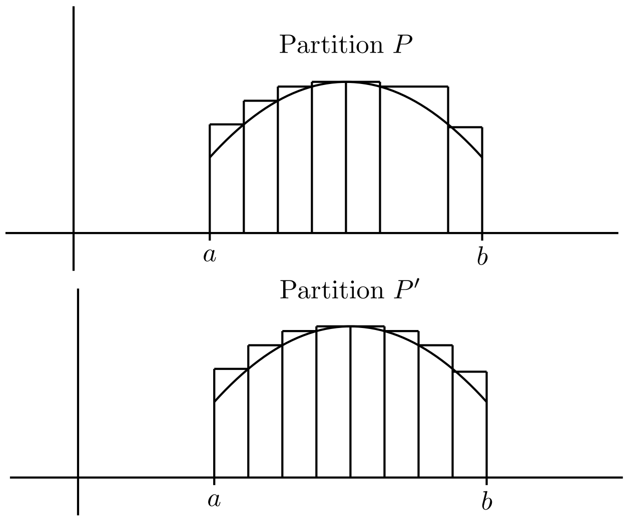
Lower Sums:



$$L(f, P') \geq L(f, P)$$

Upper Sums:



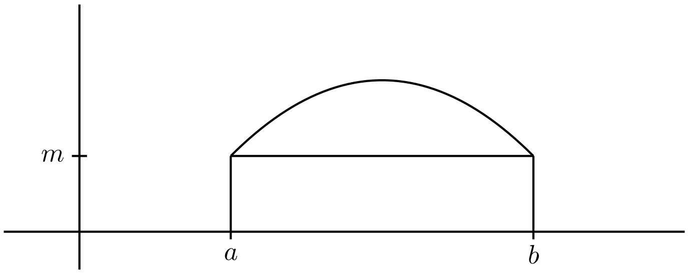
$$U(f, P') \leq U(f, P)$$

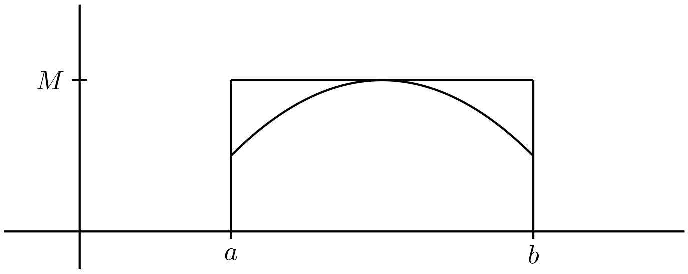
It is easy to see (note the diagrams above) that adding a single point to a partition will cause the lower sum to increase and similarly will cause an upper sum to decrease.
In general, if P' is a refinement of P we can get from P to P' by adding a single point at a time and we will find

$$L(f,P) \leq L(f,P') \text{ and } U(f,P) \geq U(f,P').$$

The partition with the least number of points is $P = \{a, b\}$.
This partition gives us a single rectangle underestimating the area under the curve and a single rectangle overestimating the area under the curve.
If we let $m = glb\{f(x) : x \in [a,b]\}$ and $M = lub\{f(x) : x \in [a,b]\}$ then

$$L(f,P) = m(b-a) \text{ and } U(F,P) = M(b-a)$$





And so we see that $m(b-a)$ is the smallest possible lower sum and $M(b-a)$ is the largest possible upper sum.

If we let

$$\mathfrak{P} = \text{ set of all possible partitions of } [a,b]$$

we have

$$L(f,P) \geq m(b-a), \text{ for all } P \in \mathfrak{P}$$

and

$$U(f,P) \leq m(b-a), \text{ for all } P \in \mathfrak{P}$$

If we let
$$\mathfrak{L} = \{L(f,P) : P \in \mathfrak{P}\}$$
and
$$\mathfrak{U} = \{U(f,P) : P \in \mathfrak{P}\}$$
then, we see from the above observations that $\mathfrak{L}$ is bounded below by $m(b-a)$ and $\mathfrak{U}$ is bounded above by $M(b-a)$.

We will now show that the sets , $\mathfrak{L}$ is also bounded above and $\mathfrak{U}$ is bounded below.

In fact we can now show *any* lower sum is less than or equal to *any* upper sum even if they are based on different partitions:

Lemma 0.1

Let f be a bounded function defined over $[a,b]$. Let P_1 and P_2 be partitions of $[a,b]$:
$$L(f,P_1) \leq U(f,P_2)$$

Proof

The partition $P_1 \cup P_2$ is a refinement of P_1 and P_2 therefore
$$L(f,P_1) \leq L(f,P_1 \cup P_2) \leq U(f,P_1 \cup P_2) \leq U(f,P_2).$$
$\square$

The above lemma says that every lower sum is smaller than any upper sum whatever and vice-versa.

Definition 0.2

If
$$\text{lub } \mathfrak{L} = \text{glb } \mathfrak{U}$$
then the function f is said to *integrable* over $[a,b]$ and the common value of lub $\mathfrak{L}$ and glb $\mathfrak{U}$ is defined to be the area under curve.
The area under the curve is denoted by $\int_a^b f$ or by $\int_a^b f(x)dx$

The dx is inserted to emphasize which symbol is the variable, e.g. $\int_a^b xy^2dx$ is different from $\int_a^b xy^2dy$

If $f(x) < 0 \ \ \forall x \in [a,b]$ then it is easy to see that $\int_a^b f$ is the negative of the area between the curve, the x-axis and the lines $x = a$ and $x = b$.
If $f(x)$ is sometimes positive and sometimes negative over $[a,b]$ it is easy to see that $\int_a^b f$ is the area under the curve above the x-axis minus the area above the curve, and below the x-axis, and the lines $x = a$ and $x = b$.

0.1.0.1 An example of a function which is ***not*** *integrable.* Not all functions are integrable. There are some very badly behaved functions which are not integrable:

Let

$$f(x) = \begin{cases} 1, x \in \mathbb{Q}, \\ 2, x \notin \mathbb{Q}. \end{cases}$$

It easy to show that this function is not integrable over any partition $[a,b]$.

For example, let us show that f is not integrable over [0,1]:

Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of [0,1] then

$$m_i = \text{glb } \{f(x) : x \in [x_i, x_{i+1}]\} = 1$$

and

$$M_i = \text{lub } \{f(x) : x \in [x_i, x_{i+1}]\} = 2.$$

therefore:

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 1.(x_i - x_{i-1}) = 1$$

$$U(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 2.(x_i - x_{i-1}) = 2\sum_{i=1}^{n}(x_i - x_{i-1}) = 2$$

$\sum_{i=1}^{n}(x_i - x_{i-1}) = 1$ because it is the sum of the lengths of all the sub-intervals into which [0,1] is sub-divided.

Therefore $\mathfrak{L} = \{1\}$ and $\mathfrak{U} = \{2\}$
and we see that lub $\mathfrak{L} \neq$glb $\mathfrak{U}$ and f is not integrable over $[a,b]$

0.1.0.2 All continuous functions are integrable..

Theorem 0.3

If f is continuous over $[a, b]$ then f is integrable over $[a, b]$.

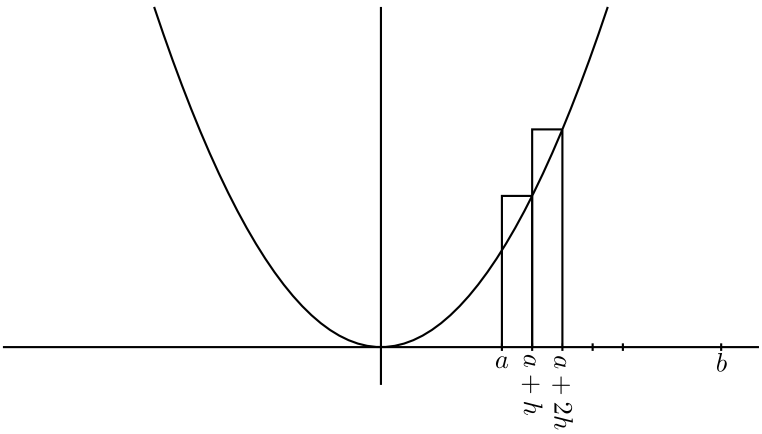
Example 0.4

Evaluate $\int_a^b x^2 \mathrm{d}x$.

Let $0 < a < b$. Let P_n be the partition of $[a, b]$ into n sub-intervals of equal length.
Each sub-interval will therefore be of length $h = \dfrac{b-a}{n}$.

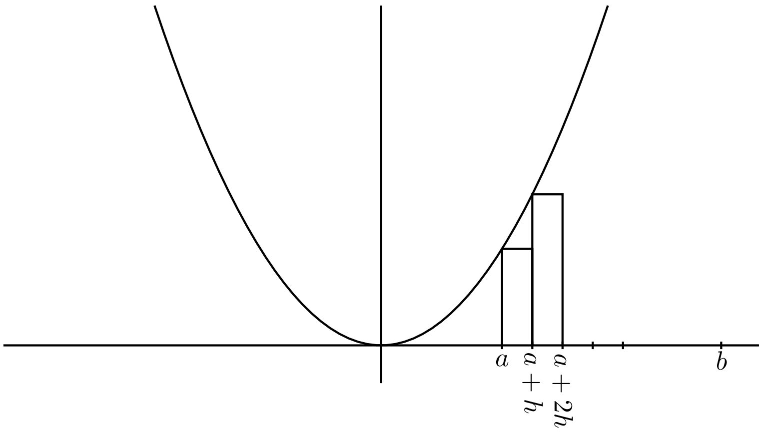
Therefore

$$P_n = \{a, a+h, a+2h, a+3h, \ldots, a+(n-1)h, a+nh = b\}$$



Note that the heights of the largest rectangles are given by

$M_1 = (a+h)^2,\ M_2 = (a+2h)^2,\ M_3 = (a+3h)^2,\ M_4 = (a+4h)^2 \ldots\ M_n = (a+nh)^2.$



the heights of the smaller rectangles are given by

$m_1 = a^2,\ m_2 = (a+h)^2,\ m_3 = (a+2h)^2,\ m_4 = (2+3h)^2 \ldots\ m_n = (a+(n-1)h)^2$

Therefore

$$U(f, P_n) = (a+h)^2h + (a+2h)^2h + (a+3h)^2h + \cdots + (a+nh)^2h = \sum_{i=1}^{n}(a+ih)^2h$$

and

$$L(f, P_n) = (a^2)h + (a+h)^2h + (a+2h)^2h + (a+3h)^2h + \cdots + (a+(n-1)h)^2h = \sum_{i=0}^{n-1}(a+ih)^2h$$

Therefore

$$U(f, P_n) = \sum_{i=1}^{n}(a+ih)^2h = \sum_{i=1}^{n}(a^2 + 2ahi + h^2i^2)h = \sum_{i=1}^{n}a^2h + 2ah^2i + h^3i^2$$

$$= \sum_{i=1}^{n}a^2h + 2ah^2\sum_{i=1}^{n}i + h^3\sum_{i=1}^{n}i^2 = na^2h + 2ah^2\frac{n(n+1)}{2} + h^3\frac{n}{6}(n+1)(2n+1)$$

$$= na^2\frac{(b-a)}{n} + 2a\frac{(b-a)^2}{n^2}\frac{n(n+1)}{2} + \frac{(b-a)^3}{n^3}\frac{n}{6}(n+1)(2n+1)$$

$$= a^2(b-a) + a(b-a)^2\frac{n(n+1)}{n^2} + (b-a)^3\frac{n(n+1)(2n+1)}{n^3}$$

$$\to a^2(b-a) + a(b-a)^2 + (b-a)^3\frac{1}{3} = \frac{1}{3}b^3 - \frac{1}{3}b^3 \text{ as } n \to \infty.$$

That is

$$\lim_{n\to\infty} U(f, P_n) = \frac{1}{3}b^3 - \frac{1}{3}b^3.$$

Similary we can show that

$$\lim_{n\to\infty} L(f,P_n) = \frac{1}{3}b^3 - \frac{1}{3}b^3.$$

Now since

$$L(f,P_n) \le \int_a^b x^2 \mathrm{d}x \le U(f,P_n)$$

it follows that

$$\lim_{n\to\infty} L(f,P_n) \le \int_a^b x^2 \mathrm{d}x \le \lim_{n\to\infty} U(f,P_n)$$

That is,

$$\frac{1}{3}b^3 - \frac{1}{3}b^3 \le \int_a^b x^2 \mathrm{d}x \le \frac{1}{3}b^3 - \frac{1}{3}b^3 \Rightarrow \int_a^b x^2 \mathrm{d}x = \frac{1}{3}b^3 - \frac{1}{3}b^3$$